

Irreducibles in the Integers modulo n

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1 Introduction

In 2011, Anderson and Frazier introduced a general theory of factorization of elements in integral domains [1]. Given a relation τ on an integral domain D , they defined a τ -factorization of an element $a \in D$ by $a = \lambda a_1 \dots a_n$ where λ is a unit in D and $a_i \tau a_j$ for all i, j . They briefly investigated the irreducible and prime elements of the integers under the congruence modulo n relation (denoted τ_n). This paper further investigates the irreducible integers under the τ_n relation for particular values of n in the hopes of finding a general form in which to express them. We were successful in finding all irreducible integers under the τ_n relation for $n = 2, 3, 4, 5, 6, 7$, and 11 by making use of another equivalence relation based on τ_n , and we were able to find a general form for all these irreducibles in Theorem 17.

2 Buildup

We begin with a few definitions for clarity.

Definition 1. Two integers x and y are said to be τ_n -related, denoted $x \tau_n y$, if, and only if, $x \equiv y \pmod{n}$.

Definition 2. For an integer x , $x = \lambda a_1 a_2 \dots a_k$ is called a τ_n -factorization of x if $\lambda \in U(\mathbb{Z})$ and $a_i \tau_n a_j$ for all i, j . We say the τ_n -factorization is proper if $k > 1$.

Definition 3. If a proper τ_n -factorization of an integer x does not exist, x is a τ_n -atom.

Definition 4. A positive integer x is a τ_n -prime if, whenever x divides a τ_n -factorization $\lambda a_1 \dots a_k$, then x divides a_i for some i .

To avoid confusion, we shall use the term “usual prime” when referencing the standard idea of prime numbers in the integers. It is worth mentioning that, clearly, all of the usual primes are τ_n -primes.

An example is in order at this point to ensure understanding.

Example 1. Consider the integer $98 = 2 * 7 * 7$. Below are 3 possible factorizations of 98:

$$\begin{aligned} &2 * 7 * 7 \\ &2 * 49 \\ &-14 * -7 \end{aligned}$$

Since $-7\tau_7 = 14$, then $-14 * -7$ is a τ_7 -factorization of 98. However, none of these factorizations are τ_2 -factorizations, and indeed none exist; thus 98 is a τ_2 -atom. This warrants the question: is 98 a τ_2 -prime? Notice that $196 = 14 * 14$, which is clearly a τ_2 -factorization, and $98|196$, but $98 \nmid 14$: thus, 98 is not a τ_2 -prime. 14, however, is a τ_2 -prime: notice that if 14 divides some τ_2 -factorization $p_1 p_2 \dots p_k$, then 2 must also divide it, and so $p_i \tau_2 0$ for some i . Since $p_i \tau_2 p_j$ for all i, j , then each term in this product must be τ_2 -related to 0; that is, they are all divisible by 2. Further, since 14 divides the product, then 7 must also divide it, and since 7 is a usual prime it must divide some p_m . Since both 2 and 7 divide this p_m , then $14|p_m$, and so 14 is a τ_2 -prime.

This example illustrates that some τ_n -atoms may not be τ_n -primes. However, τ_n -primes are all, in fact, τ_n -atoms; this is shown in [1]. An alternative proof is given below.

Theorem 5. *If an integer x is a τ_n -prime, then x is a τ_n -atom.*

Proof. Let integer x be a τ_n -prime. Then if x divides some τ_n -factorization $\pm p_1 p_2 \dots p_k$, $x|p_i$ for some i . Suppose, by way of contradiction, that x is not a τ_n -atom. Then there must exist some proper τ_n -factorization for x ; denote it by $\pm x_1 x_2 \dots x_m$. Notice, then, that $x|\pm x_1 x_2 \dots x_m$. Hence, $x|x_j$ for some j . However, since $x_1 x_2 \dots x_m$ is a *proper* τ_n -factorization, then necessarily $x_j < x$ for all j , and so $x \nmid x_j$ for all j . A contradiction arises; thus, x must be a τ_n -atom. \square

In their paper [1], Anderson and Frazier explored the τ_2 -atoms and τ_2 -primes in particular. They were able to show that the τ_2 -primes were of the form $2p$ for usual prime $p \neq 2$, and that the τ_2 -atoms were of the form $2p_1 p_2 \dots p_k$ for usual primes $p_i \neq 2$.

Anderson and Frazier were also able to find a general form for the τ_n -primes for any value of n :

Theorem 6. *An integer b is a τ_n -prime if, and only if, $b = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k} q$, where $e_i = 1$ or 0 for all i , p_j is a usual prime which divides n for all j , and q is a usual prime which does not divide n or $q = 1$.*

While this result is quite good, the τ_n -primes are but a subset of the τ_n -atoms. Thus, we investigated the τ_n -atoms for different values of n in the interest of finding a general form, similar to that for τ_n -primes in Theorem 6.

3 The τ_3 -atoms, τ_4 -atoms, and τ_6 -atoms

It is logical to start with the next least complicated structure, that of the integers under the τ_3 -relation. We already know that the usual primes are τ_n -atoms for all values of n by Theorem 5, but (as was the case with the τ_2 -atoms) there are almost certainly more. For example, we know by Theorem 6 that $3p$ is a τ_3 -prime (and thus a τ_3 -atom) for all usual primes p . However, when investigating the τ_3 -atoms, an interesting complication arose: that of factorizations involving negative factors. Since $U(\mathbb{Z}) = \{\pm 1\}$, then any number of factors can be negated merely by introducing the appropriate number of factors of -1 . For example, consider the proper factorizations of 28. It may be tempting to assume that only 3 exist (namely, $2*2*7$, $4*7$, and $2*14$), but in fact 13 possible proper factorizations exist by simply

negating different combinations of factors. This complication, however, is easily solved by recalling that $-k\tau_n(n-k)$ for all k, n . Thus, if an integer x is τ_3 -related to 2, then $-x\tau_3 1$. With this in mind, we can show the following:

Theorem 7. *The τ_3 -atoms consist of the usual primes and integers of the form $3p_1p_2\dots p_m$, where p_i is a prime not equal to 3 for all i .*

Proof. By Theorem 5, we know that the usual primes are τ_3 -atoms.

Let k be a τ_3 -atom with usual prime factorization $p_1p_2\dots p_m$, $m > 1$. We shall proceed by considering cases based on the multiplicity of 3 in k .

Case 1: Suppose 3 does not divide k . Then for all i , $p_i\tau_3 1$ or $p_i\tau_3 2$. Suppose that b of the p_i terms are τ_3 -related to 1 and $m-b$ of the p_i terms are τ_3 -related to 2. Then if $m-b$ is even, simply negating all the p_i terms τ_3 -related to 2 will yield a τ_3 -factorization; similarly, if $m-b$ is odd, then negate all the p_i terms τ_3 -related to 2 and introduce a factor of the unit -1 , yielding a τ_3 -factorization. Either way, k has been shown to have a τ_3 -factorization, yet k is a τ_3 -atom by hypothesis - a contradiction. Thus, 3 must divide k .

Case 2: Suppose 3 divides k twice or more. Then the usual prime factorization of k is of the form $3 * 3p_3\dots p_m$. But $3 * (3p_3\dots p_m)$ is a τ_3 -factorization of k , since both factors are τ_3 -related to 0; thus, k is not a τ_3 -atom, giving rise to another contradiction. Thus, 3 must divide k exactly once. It only remains to be shown that any integer for which this holds is indeed a τ_3 -atom.

Case 3: Suppose 3 divides k exactly once. Then the usual prime factorization of k is of the form $3p_2\dots p_m$, where $p_i \neq 3$ for all i . Notice that, in any grouping of these factors, whichever factor is divisible by 3 must be τ_3 -related to 0, while all the other factors cannot possibly be τ_3 -related to 0, since they are necessarily not divisible by 3. Thus, no τ_3 -factorization for k exists, and so k is a τ_3 -atom. \square

Notice that the τ_3 -atoms are quite similar to the τ_2 -atoms: we know the τ_2 -atoms (other than the usual primes) are of the form $2p_1p_2\dots p_k$ where p_i is a prime not equal to 2, while the above theorem shows that the τ_3 -atoms are of the form $3q_1q_2\dots q_k$ where q_i is a prime not equal to 3. At this point, one cannot help but wonder whether an integer of the form $np_1p_2\dots p_k$ where n does not divide $p_1p_2\dots p_k$ is a τ_n -atom for any n . This is not always the case (consider that 8 divides 16 exactly once, as desired, yet $4 * 4$ is a τ_8 -factorization of 16), but it *does* always hold when n is a usual prime.

Theorem 8. *For a positive prime integer n , any integer of the form $np_1p_2\dots p_k$ where n does not divide $p_1p_2\dots p_k$ (that is, an integer in which the multiplicity of n is exactly 1) is a τ_n -atom. If the multiplicity of n is greater than 1 in any integer, then that integer must not be a τ_n -atom.*

Proof. Let x be an integer with prime factorization $np_1p_2\dots p_k$ where n does not divide $p_1p_2\dots p_k$. Since n is a usual prime, then, similar to Case 3 of the proof of Theorem 7, it can be seen that in any grouping of the factors of x , one factor (namely, the one which n divides) must be τ_n -related to 0, while the others necessarily must not be τ_n -related to 0. Thus, no τ_n -factorization of x exists, and so x must be a τ_n -atom.

Let y be an integer in which the multiplicity of n is at least 2. Then, as in Case 2 of Theorem 7, it can be seen that we may simply take the τ_n -factorization $n * (y/n)$, both of

which must be τ_n -related to 0, and so y is not a τ_n -atom. Notice that n need not be prime for this to be true. \square

Because of how nice these properties are, we will allow n to be a usual prime for the remainder of the paper unless stated otherwise. Much insight can be gained into the τ_n -atoms for composite values of n based on this, however, given the following result:

Theorem 9. *If an integer x is a τ_n -atom, and $n|m$ for some integer $m \geq n$, then x is a τ_m -atom.*

Proof. Let n, m be positive integers such that $n|m$ and let x be a τ_n -atom. Then in any factorization $x = \pm a_1 a_2 \dots a_k$, there must be some factors a_i and a_j such that $a_i \not\sim_n a_j$; that is, $n \nmid a_i - a_j$. Then since $n|m$, $m \nmid a_i - a_j$, and so $a_i \not\sim_m a_j$. Thus there must be no τ_m -factorization of x , and so x is a τ_m -atom. \square

This actually allows us to find all the τ_4 - and τ_6 -atoms.

Theorem 10. *An integer x is a τ_4 -atom if, and only if, x is a τ_2 -atom.*

Proof. We know by Theorem 9 that a τ_2 -atom is a τ_4 -atom. Suppose, by way of contradiction, that an integer x is a τ_4 -atom and not a τ_2 -atom. Recall that the τ_2 -atoms are the usual primes and integers in which the multiplicity of 2 is exactly one. Then there are two cases.

Case 1: 2 divides x more than once; denote this by $x = 2^{j+1} p_1 p_2 \dots p_k$, where p_i is an odd prime for all i . Notice that 2 multiplied by any odd number returns an even value not divisible by 4; that is, an integer τ_4 -related to 2. Thus, we may simply write x as $2 * 2 * \dots * (2 p_1 p_2 \dots p_k)$, where there are j factors of 2 before the final factor. Then all these factors are τ_4 -related to 2, and thus we have a τ_4 -factorization for x , which produces a contradiction.

Case 2: 2 does not divide x . Then all prime factors of x must be odd, and so must all be τ_4 -related to either 1 or 3. Notice that $-1 \tau_4 3$; thus, as in Case 1 of Theorem 7, we can simply negate all the factors of x that are τ_4 -related to 1 and, if necessary, include a factor of -1 to produce a τ_4 -factorization of x , producing a contradiction.

Therefore any τ_4 -atom must also be a τ_2 -atom. \square

Theorem 11. *An integer x is a τ_6 -atom if, and only if, x is either a τ_2 -atom or a τ_3 -atom.*

Proof. Again, by Theorem 9 we know that the τ_2 - and τ_3 -atoms must be τ_6 -atoms. Suppose, by way of contradiction, that an integer x is a τ_6 -atom that is neither a τ_2 -atom nor a τ_3 -atom; that is, the multiplicity of neither 2 nor 3 in x is exactly 1, and x is not a usual prime. The following 4 cases exhaust all possibilities, then:

Case 1: The multiplicities of both 2 and 3 in x is zero; that is, $x = p_1 p_2 \dots p_k$ where p_i is a usual prime not equal to 2 or 3 for all i . Consider some arbitrary usual prime p_j which divides x . Clearly $p_j \not\sim_6 0$, else $6|p_j$ and so p_j is not a usual prime. Nor can $p_j \tau_6 2$ be true, else $p_j = 6q + 2$ for some integer q , and so $2|p_j$ and, since $p_j \neq 2$, p_j is not a usual prime. Similarly, $p_j \not\sim_6 3$ and $p_j \not\sim_6 4$. Thus, $p_j \tau_6 1$ or $p_j \tau_6 5$, and since p_j is arbitrary, this holds for all j . Notice, however, that $-1 \tau_6 5$, and so, similar to Case 1 of Theorem 7, we may simply negate all usual prime factors of x and, if necessary, introduce a factor of -1 to produce a τ_6 -factorization of x . Thus, either 2 or 3 must have multiplicity at least 1 in x , and since their multiplicity cannot be exactly one by hypothesis, it must be greater than one.

Case 2: The multiplicity of 2 in x is greater than 1, and the multiplicity of 3 in x is zero; that is, $x = 2^y p_1 p_2 \dots p_k$ where p_i is a usual prime not equal to 2 or 3 for all i and $y > 1$. The product of any number of the p_i factors cannot be τ_6 -related to 3, else that product would be equal to $6q + 3$ for some integer q , and thus would necessarily be divisible by 3; but $3 \nmid x$. Further, by the same logic as Case 1, none of the p_i factors can be τ_6 -related to either 2, 4, or 6, else they must not be usual primes. Thus, they must all be τ_6 -related to either 1 or 5. Notice that the product of all the p_i factors τ_6 -related to 1 is still τ_6 -related to 1; thus, take their product and call this product d_1 . Now, since $5\tau_6 = 1$, then the product of all the d_i factors τ_6 -related to 5 must be τ_6 -related to either 1 or -1 ; take this product and call it x_5 . Then $x = 2^y x_1 x_5$. If $x_5 \tau_6 1$, then notice that $x = 2 * 2 * \dots * (2d_1 d_5)$ is a τ_6 -factorization, since $d_1 d_5 \tau_6 1$, and so $2d_1 d_5 \tau_6 2$. If $d_5 \tau_6 -1$, then, similarly, $x = 2 * 2 * \dots * (-2d_1 d_5) * -1$ is a τ_6 -factorization, since -1 is a unit.

Case 3: The multiplicity of 3 in x is greater than 1, and the multiplicity of 2 in x is zero; that is, $x = 3^y p_1 p_2 \dots p_k$ where p_i is a usual prime not equal to 2 or 3 for all i and $y > 1$. The proof of this case is similar to that of Case 2.

Case 4: The multiplicities of both 2 and 3 in x are greater than 1. Then $x = 2^y 3^z p_1 p_2 \dots p_k$, where p_i is a usual prime not equal to 2 or 3 for all i and $y, z > 1$. Then notice that $x = 6 * (2^{y-1} 3^{z-1} p_1 p_2 \dots p_k)$ is a τ_6 -factorization of x .

Thus, x must be either a τ_2 -atom or τ_3 -atom. □

4 The μ_n relation

For higher values of n , determining the τ_n -atoms becomes significantly more difficult, in part due to the issue of negative factors. For example, 6 may seem to be a τ_5 -atom, since $6 = 2 * 3$ and $2 \nmid_5 3$, but notice that $6 = -1 * -2 * 3$, and since -1 is a unit and $-2\tau_5 3$, then this is a τ_5 -factorization of 6. This problem becomes significantly more difficult to overcome when considering integers with very many factors. We will solve this problem by introducing the μ_n relation.

Following the observation that $-k\tau_n(n - k)$, we write the following definition:

Definition 12. For two integers x and y , x is μ_n -related to y , denoted $x\mu_n y$, if $x\tau_n \pm y$.

In this way, we can simply worry about whether a factorization exists in which all terms are μ_n -related, eliminating the need to consider the unit -1 .

Since the μ_n relation is based on the τ_n relation, which we know is an equivalence relation, it is of interest whether μ_n is an equivalence relation as well, and it should be rather clear to the reader that it is. As a result, we can consider the equivalence classes of the integers under the μ_n relation, which we will denote (at this time) using the familiar notation $[x] = \{y \in \mathbb{Z} | y\mu_n x\}$. Notice that, as there are n equivalence classes of the τ_n relation, there must be $\lceil n/2 \rceil$ equivalence classes of the μ_n relation, and since we are only concerned with when n is a usual prime, $\lceil n/2 \rceil = \frac{n+1}{2}$.

Theorem 13. For a usual prime $n > 2$, and for any positive integer a such that $1 < a < n$, $\left\{ [0], [a], [a^2], \dots, [a^{\frac{n-1}{2}}] \right\}$ is the set of all equivalence classes of the integers under the μ_n relation. Further, $a^{\frac{n-1}{2}} \mu_n 1$.

Proof. Since n is a usual prime, it is relatively prime to any $a < n$. Thus, by Fermat's Little Theorem, we can say that, for such an $a < n$, $\{[0], [a], [2a], \dots, [\frac{n-1}{2}a]\}$ is a set of $\frac{n+1}{2}$ distinct equivalence classes of the integers under the τ_n -relation. We claim that these are also distinct equivalence classes of the integers under the μ_n -relation. Suppose not: that is, suppose $sa\mu_nra$ for some nonnegative integers $r, s \leq \frac{n-1}{2}$. Then $sa\tau_n \pm ra$, and so $n|(s \pm r)a$. Since $0 < a < n$, then $n \nmid a$; thus $n|s \pm r$. Notice that $|s \pm r| \leq n-1$, and thus $n|s \pm r$ if, and only if, $s \pm r = 0$. Then $s = \pm r$, and since s and r are nonnegative then $s = r$. Thus, the aforementioned equivalence classes are indeed distinct in the integers under the μ_n relation, and further, since there are $\frac{n+1}{2}$ such classes, these must represent *all* of the equivalence classes of the integers under the μ_n relation.

Notice that the set $S = \{1, 2, \dots, \frac{n-1}{2}\}$ has a representative from each equivalence class except $[0]$. Then the product of all the elements of S must be μ_n -related to the product $\{b, 2b, \dots, \frac{n-1}{2}b\}$ for any positive integer $b < n$; that is, $(\frac{n-1}{2})!\tau_n \pm (\frac{n-1}{2})! * b^{\frac{n-1}{2}}$. Since $\frac{n-1}{2}!$ is relatively prime to n , then it must have an inverse modulo n . Thus, $\pm 1\tau_nb^{\frac{n-1}{2}}$. Since b was arbitrary, this holds for any positive integer less than n .

Consider the set $\{[0], [a], [a^2], \dots, [a^{\frac{n-1}{2}}]\}$ for some arbitrary positive integer $a < n$. It will be shown that this, too, is the set of all distinct equivalence classes of the integers under the μ_n relation. Clearly none of the classes represented by the powers of a must be equivalent to that represented by $[0]$, since n is a usual prime. Suppose that $a^s\mu_na^r$ where $r < s \leq \frac{n-1}{2}$. Then $a^{s-r}\mu_n1$, so $a^{s-r}\tau_n \pm 1$. Suppose that $(s-r)$ is the smallest positive integer such that $a^{s-r}\tau_n \pm 1$. By the Quotient Remainder Theorem, $\frac{n-1}{2} = (s-r)q + k$ for some positive integers q and k , with $k < (s-r)$. Then since, as stated in the previous paragraph, $a^{\frac{n-1}{2}}\tau_n \pm 1$, then consider that $\pm 1\tau_na^{\frac{n-1}{2}}\tau_na^{(s-r)q+k} = a^{(s-r)q}a^k\tau_n(a^{(s-r)})^qa^k\tau_n(\pm 1)^qa^k\tau_n \pm a^k$; in short, $\pm 1\tau_na^k$. But $k < (s-r)$, a contradiction. Thus, $sa \not\mu_nra$, and so the equivalence classes must be distinct. \square

For any usual prime n , notice that if an integer x has usual prime factorization $x = p_1p_2\dots p_k$, and $p_i\mu_n0$ for any i , then we already know by Theorem 8 whether x is a τ_n -atom: if only one factor is in the μ_n equivalence class $[0]$ then it is a τ_n -atom, and if more than one is, it is not. Thus, for the remainder of the paper we will only be concerned with integers whose usual prime factorizations contain no factors μ_n -related to 0.

At this point, we will introduce a new notation to further simplify the interpretation of usual prime factorizations of integers in the context of τ_n -factorizations by indexing the equivalence classes of the μ_n relation. First, let y and z be arbitrary integers with $z\mu_n1$. Clearly $y \in [y]$ and $z \in [z]$, but consider yz . Notice that, since $z\mu_n1$, then $yz\mu_ny$. In essence, when we multiply integers by elements of $[1]$, we do not change equivalence classes. We shall denote an arbitrary element of $[1]$ by x_0 . This notation will become clearer in time.

Now we know by Theorem 13 that for any integer a such that $1 < a < n$ where n is a usual prime, $a^{\frac{n-1}{2}}\mu_n \pm 1$. We will denote elements of the equivalence class $[a]$ by x_1 , of $[a^2]$ by x_2 , and so on. This indexing is not unique to each usual prime n , but relies only on a choice of the integer a for that value of n . Notice that, for a given usual prime n , these indexing values will range from 0 to $\frac{n-3}{2}$.

We shall now demonstrate the helpful nature of this indexing system we have built up. Suppose that we are curious about whether some integer x is a τ_n -atom for some

large usual prime n , and x has a usual prime factorization $x = x_1x_2x_3$; that is, for some integer $a \in (1, n)$, the usual prime factorization of x has one factor which is μ_n -related to a , one which is μ_n -related to a^2 , and one μ_n -related to a^3 . Now, clearly, $x_1x_2x_3$ is not a τ_n -factorization of x . However, consider the product x_1x_2 . Since $x_1\mu_na$ and $x_2\mu_na^2$, then $x_1x_2\mu_na^2a = a^3$. But $x_3\mu_na^3$, so $x_1x_2\mu_nx_3$. Then $x_1x_2\tau_n \pm x_3$, and so x has a τ_n -factorization; namely, either $x = (x_1x_2) * x_3$ or $x = -1 * (x_1x_2) * x_3$. Notice, though, that $x_1x_2\mu_nx_3$, and (looking at the indices) $1 + 2 = 3$. This is done to create an easier, addition-based indexing which allows us to combine terms in an easy and predictable manner. In general, in fact, $x_ix_j\mu_nx_{i+j \bmod \frac{n-1}{2}}$, which should be clear since $x_ix_j\mu_na^ia^j = a^{i+j}\mu_nx_{i+j}$; the "mod" clause is simply inserted to ensure that we use the smallest representative, as $a^{\frac{n-1}{2}}\mu_n1$. We shall approach multiplication of these representatives by considering their indices under addition mod $\frac{n-1}{2}$. We will adopt one last convention of notation for the remainder of the paper: the factorization of x mentioned earlier, $x = (x_1x_2) * x_3$, will be written as $x = x_3 * x_3$. This is not intended to imply that there is a single factor with a multiplicity of 2 in x , but instead to merely state that there are 2 factors in x which are both in the μ_n equivalence class $[a^3]$ for some integer $a \in (1, n)$. Henceforth, it should not be assumed that all integers denoted x_i are equal for a particular integer i , but instead that both are merely within the same μ_n equivalence class.

We shall use the results of this section to investigate the τ_n -atoms for higher values of n . The next section is appropriately shorter than the previous sections, as this new notation streamlines the process of finding τ_n -atoms greatly.

5 The τ_5 -atoms

The next usual prime after 3 is 5, and so we will naturally move on at this point to investigate the τ_5 -atoms. By Theorem 8, we know that integers of the form $5p_1p_2\dots p_k$, where p_i is a usual prime not equal to 5, are τ_5 -atoms, along with the usual primes. However, given the example at the beginning of the previous section, 6 is also a τ_5 -atom, yet meets neither of these criteria, and so we aim to characterize the remaining τ_5 -atoms. By Theorem 8, we can ignore any other integers which are divisible by 5; thus, for the remainder of the section we only consider those integers not divisible by 5 unless otherwise stated. Notice that there are only two μ_5 equivalence classes other than $[0]$: $[1]$ and $[2]$; thus, we will denote elements of $[1]$ by x_0 and elements of $[2]$ by x_1 .

Clearly any usual prime factorization of an integer with no x_1 factors must contain only x_0 elements, and so such an integer is not a τ_5 -atom. Similarly, we need not consider those integers whose usual prime factorizations have no x_0 factors. There must be some x_0 and some x_1 factors.

Theorem 14. *The τ_5 -atoms are the usual primes, integers whose usual prime factorizations are of the form $5p_1p_2\dots p_k$ where p_i is a usual prime not equal to 5, and integers whose usual prime factorizations are of the form $x_0 * x_0 * \dots * x_0 * x_1$, where each x_0 is a usual prime in the μ_n equivalence class $[1]$ and x_1 is a usual prime in the μ_n equivalence class $[2]$.*

Proof. It suffices to show that integers with prime factorizations of the form $x_0 * x_0 * \dots * x_0 * x_1$ are τ_5 atoms, and that no other integers which are neither usual primes nor divisible by 5 are

τ_5 -atoms. Recall, again, that all factors denoted x_0 are *not necessarily equal*; this notation merely communicates the μ_n equivalence class of each factor.

Let x be an integer with usual prime factorization $x = x_0 * x_0 * \dots * x_0 * x_1$. First, note that this is not a τ_5 -factorization. Further, since $0 + 0 = 0$, the product of any number of x_0 factors is simply another x_0 factor, and since $0 + 1 = 1$, then the product $x_0 * x_1$ is another x_1 factor. Thus, there must always be exactly one x_1 factor in any factorization of x , and so x must not have a τ_5 -factorization, as there must either be only x_0 factors or multiple x_1 factors in a τ_5 -factorization.. Thus, x is a τ_5 -atom.

To show that there are no overlooked τ_5 -atoms, let y be an integer with usual prime factorization $y = x_0 * x_0 * \dots * x_0 * x_1 * x_1 * \dots x_1$; that is, any integer which is neither a usual prime nor divisible by 5, and which has more than one x_1 factor. Then since $0 + 0 = 0$, we may simply multiply all of the x_0 factors together into a single x_0 factor; that is, $y = x_0 * x_1 * x_1 * \dots * x_1$. Then, since $0 + 1 = 1$, multiplying the x_0 factor by a single x_1 factor produces another x_1 factor, and so $y = (x_0 * x_1) * x_1 * \dots * x_1 = x_1 * x_1 * \dots * x_1$. Thus, y has a τ_n -factorization, since it has a factorization in which all factors share a μ_n equivalence class. \square

This exhausts all possibilities. A τ_5 -atom that is neither a usual prime nor divisible by 5 must have at least one x_1 factor, but it cannot have more than one; thus it must have exactly one. It must have at least one x_0 factor (else it is a usual prime, a contradiction), however it may have any number of x_0 factors in addition to the obligatory x_1 factor.

6 The τ_7 -atoms

Next, we move on to the τ_7 -atoms. Again, we know by Theorem 8 that integers of the form $7p_1p_2\dots p_k$ where p_i is a usual prime not equal to 7 are τ_7 -atoms, along with the usual primes. Notice that there are 3 nonzero equivalence classes of the μ_7 relation; thus, we shall be calling their representatives x_0, x_1 , and x_2 , similar to the way we denoted the μ_5 equivalence classes.

Recall that by Theorem 14, integers whose usual prime factorizations are of the form $x_0 * x_0 * \dots * x_0 * x_1$ are τ_5 -atoms (where x_0 and x_1 refer to μ_5 equivalence classes). When we consider this kind of integer in reference to μ_7 equivalence classes, we have two possibilities: $x_0 * x_0 * \dots x_0 * x_1$ or $x_0 * x_0 * \dots x_0 * x_2$. It just so happens that integers with usual prime factorizations of either of these forms are τ_7 -atoms; the proof of Theorem 14 suffices to show this point.

However, there are other τ_7 -atoms that meet none of these criteria. Consider that 6 meets none of these criteria, yet is clearly a τ_7 -atom, as $2 \not\mu_7 3$. We wish to characterize the remaining τ_7 -atoms.

Theorem 15. *The τ_7 -atoms are the usual primes, along with integers whose usual prime factorizations are of the form $7p_1p_2\dots p_k$ where p_i is a usual prime not equal to 7, or integers whose usual prime factorizations can be expressed as follows:*

- $x_0 * x_0 * \dots * x_0 * x_1$

- $x_0 * x_0 * \dots * x_0 * x_2$

- $x_1 * x_2$

where x_0, x_1 , and x_2 are factors from the μ_7 equivalence classes [1], [2], and [3], respectively.

Proof. All these cases have been addressed except integers with usual prime factorizations of the form $x_1 * x_2$, yet it is evident that such an integer must not have a τ_7 -factorization, since both factors are already usual primes. Thus, such integers must be τ_7 -atoms.

It must be shown that the criteria listed in the theorem exhaust all possibilities. Thus we consider the following cases, which exhaust all possibilities:

Case 1: Let x be an integer with usual prime factorization of the form $x_0 * x_0 * \dots * x_0 * x_1 * x_1 * \dots * x_1 * x_2 * x_2 * \dots * x_2$, where there is at least one x_0 factor, at least one x_1 factor, and at least one x_2 factor. Notice that all x_0 factors can be multiplied together into a single x_0 factor, since $0 + 0 = 0$, and so we consider x to be of the form $x_0 * x_1 * x_1 \dots * x_1 * x_2 * x_2 * \dots * x_2$ for the remainder of this case.

Suppose that there are an even number of x_1 factors. Then since $1 + 1 = 2$, multiply the x_1 factors together in pairs to make half as many x_2 factors. Then $x = x_0 * x_2 * x_2 \dots * x_2$. Now, since $0 + 2 = 2$, then notice that $x = (x_0 * x_2) * x_2 * \dots * x_2 = x_2 * x_2 \dots * x_2$, which means x has a τ_7 -factorization; thus x is not a τ_7 -atom.

Suppose that there are an even number of x_2 factors. Then since $2 + 2 = 1 \pmod{3}$ and $3 = \frac{7-1}{2}$, then by Theorem 13 the product $x_2 * x_2$ must result in an x_1 factor. Thus, similar to the previous paragraph, we see that x has a τ_7 -factorization of all x_1 factors, and so is not a τ_7 -atom.

Suppose that there are an odd number of both x_1 and x_2 factors. If there are the same number of x_1 and x_2 factors, then simply multiply each x_1 factor by an x_2 factor will yield an x_0 factor and so a τ_7 -factorization of x exists. Suppose, instead, that there are more x_1 factors than x_2 factors. Then there must be at least two more x_1 factors than x_2 factors, since there are an odd number of both. Thus, following the same process of multiplying each x_2 factor by an x_1 factor until none remain, we shall see that $x = x_0 * x_0 * \dots * x_0 * x_1 * x_1 \dots * x_1$. Since we can multiply all x_0 factors together and then multiply again by a single x_1 factor to yield an x_1 factor (as $0 + 0 + \dots + 0 + 1 = 1$), we see that x has a τ_7 -factorization of all x_1 factors. If we assume that there are more x_2 factors than x_1 factors, this same process will provide a τ_7 -factorization of x with only x_2 factors instead.

Thus, an integer with a usual prime factorization of the form in this case cannot be a τ_7 -atom.

Case 2: Let x be an integer with usual prime factorization of the form $x_0 * x_0 * \dots * x_0 * x_i * x_i * \dots * x_i$ for $i = 1$ or $i = 2$, where there are at least two x_i factors. Then since $0 + 0 + \dots + 0 + i = i$, we see that we may simply multiply all x_0 factors and one x_i factor to yield another x_i factor. Thus, $x = x_i * x_i * \dots * x_i$, and we see that x has a τ_7 -factorization. Thus, an integer with a usual prime factorization of this form cannot be a τ_7 -atom.

Case 3: Let x be an integer with usual prime factorization of the form $x_1 * x_1 * \dots * x_1 * x_2 * x_2 * \dots * x_2$, where there are at least two x_1 factors or at least two x_2 factors. If

there are an even number of either x_1 or x_2 factors, then following the process of Case 1 when there are even numbers of x_1 or x_2 factors, we can simply multiply factors in pairs to produce factors of the other kind, and so can create a τ_7 -factorization of x . If there are an odd number of both x_1 and x_2 factors, then we can simply follow the process of Case 1 when there are odd numbers of x_1 and x_2 factors to create a τ_7 -factorization of x . Thus, integers with usual prime factorizations of this form must not be τ_7 -atoms. \square

These cases exhaust all possibilities; thus, the cases outlined in the statement of the theorem must be the only ones which are τ_7 -atoms. Our goal of finding patterns or generalizations of τ_n -atoms is being developed, and is looking more likely, as we see that forms of usual prime factorizations that are the same as those of the τ_n -atoms for lower usual prime values of n still yield τ_n -atoms for higher usual prime values of n . We shall examine one more usual prime value of n before we present our findings.

7 The τ_{11} -atoms

The next usual prime is 11, and this is the largest jump we have made thus far. Previously, we have only increased the value of n by either one or two; this time we advance by four. As a result, we also gain more μ_{11} equivalence classes: other than $[0]$, we now have $[1]$, $[2]$, $[3]$, $[4]$, and $[5]$. This makes things far more complicated. We shall denote elements of $[1]$ by x_0 , as usual, and we denote elements of $[2^i]$ by x_i where $1 \leq i \leq 4$.

As always, we know that the usual primes are τ_{11} -atoms, and we know by Theorem 8 that integers with usual prime factorizations of the form $11p_1p_2\dots p_k$ where p_i is a usual prime not equal to 11 are τ_{11} -atoms. Given our observations thus far, it is also reasonable to ask whether integers with prime factorizations of the form $x_0 * x_0 * \dots * x_0 * x_i$ where $i \neq 0$ and of the form $x_i * x_j$ where $i \neq j$ are τ_{11} -atoms, since they are τ_n -atoms for usual prime $n < 11$, and indeed we see they are, for the same reasons outlined in the previous sections. In fact, in the latter case we can even include x_0 factors as well, so long as the product $x_i * x_j$ does not yield an x_0 factor; in other words, if we have usual primes x_i and x_j such that $i + j \neq 0 \pmod{5}$, then $x_0 * x_0 * \dots * x_0 * x_i * x_j$ is a τ_{11} -atom. However, yet again we see that there must be other conditions which can be met to produce a τ_{11} -atom; it just so happens that 50 is a τ_{11} -atom, but does not meet any of the aforementioned criteria. Nor does 296, another τ_{11} -atom. In fact, the usual prime factorizations of these two integers do not even have much in common, with $50 = 2 * 5 * 5 = x_1 * x_4 * x_4$ and $296 = 2 * 2 * 2 * 37 = x_1 * x_1 * x_1 * x_2$.

Theorem 16. *The τ_{11} -atoms are the usual primes, along with integers whose usual prime factorizations are of the form $11p_1p_2\dots p_k$ where p_i is a usual prime not equal to 11, or integers whose usual prime factorizations can be expressed as follows:*

- $x_0 * x_0 * \dots * x_0 * x_i$ where $i \neq 0$
- $x_i * x_j$ where $i \neq j$
- $x_0 * x_0 * \dots * x_0 * x_i * x_j$, where $i \neq j$ and $i + j \neq 0 \pmod{5}$

- $x_i * x_i * x_j$ where $0 \neq i, j \pmod{5}$, $i \neq j$ and $2i \neq j \pmod{5}$
- $x_0 * x_0 * \dots * x_0 * x_i * x_i * x_j$, where $i \neq j$, $2i \neq j \pmod{5}$, and $2i + j \neq 0 \pmod{5}$
- $x_i * x_i * x_i * x_{2i \pmod{5}}$ where $i \neq 0 \pmod{5}$.

Proof. As a first step toward discovering all types of τ_{11} -atoms, a table was formed which detailed all possible combinations of μ_{11} equivalence class representatives (other than x_0) with at most four representatives per class. This table was manually checked, case by case, for which combinations were reducible and which were not. The table itself is not listed here due to its size: there are a total of 625 cases to check.

Of the irreducible elements, most can be readily described by the cases of the τ_7 -atoms, namely the first 5 cases listed in the statement of the theorem above (from usual primes through $x_0 * x_0 * \dots * x_0 * x_i * x_j$). At this point, then, we briefly show that the last 3 cases do indeed describe τ_{11} -atoms.

$x_i * x_i * x_j$ where $0 \neq i, j \pmod{5}$, $i \neq j$ and $2i \neq j \pmod{5}$: Since $2i \neq j \pmod{5}$, then clearly $x_{2i} * x_j$ cannot be a τ_{11} -factorization; thus, consider the only alternative: $x_{i+j} * x_i$. Suppose, by way of contradiction, that this is a τ_{11} -factorization. Then $i + j = i \pmod{5}$, and so $j = 0 \pmod{5}$. But $j \neq 0 \pmod{5}$; here arises a contradiction. Thus, an integer with a usual prime factorization of this type must be a τ_{11} -atom. The proof of the next case is clearly covered by this one with the addition of the final condition, $2i + j \neq 0 \pmod{5}$.

$x_i * x_i * x_i * x_{2i \pmod{5}}$ where $i \neq 0 \pmod{5}$: Since $ni \neq mi \pmod{5}$ for all $n \neq m$ where $0 < n, m < 5$, any arrangement of these factors clearly does not result in a τ_{11} -factorization. Thus, an integer with a usual prime factorization of this type must be a τ_{11} -atom.

Now we must show that this list of cases is exhaustive. As the initial table only considered sets of factors where there were at most 4 representatives per μ_{11} equivalence class (other than x_0), it is necessary to prove that any combination of factors involving more than 4 representatives per μ_{11} equivalence class must not be an atom. First, note that any product of 5 elements from the same μ_{11} equivalence class must yield an x_0 element, since $5i \equiv 0 \pmod{5}$. Take some arbitrary combination of μ_{11} representatives already considered (that is, one with at most 4 representatives per equivalence class other than x_0), and call its product x . Then, for each $i = 1, 2, 3, 4$, consider $y_i = x * x_i * x_i * x_i * x_i * x_i$. It is evident that, if x was not a τ_{11} -atom, then y_i must not be for any value of i , since the product of the x_i terms produces an x_0 term, which can be trivially absorbed into the τ_{11} -factorization of x . Thus, we are only concerned with the case when x is a τ_{11} -atom; that is, when the factorization of x meets any of the criteria listed in the theorem. In this case it is easy to show (again, by exhaustion) that, regardless of i , y_i is not a τ_{11} -atom; again, an explicit proof is not listed here in consideration of length. This means, then, that any factorization involving 5 or more representatives from a single μ_{11} equivalence class other than x_0 must necessarily be reducible. Thus, the cases listed in the theorem must exhaust all τ_{11} atoms. \square

It seems that as n increases, the number of cases of usual prime factorizations that result in τ_n -atoms generally increases as well. This is bothersome, and so we wish to generalize our findings thus far in the hope of finding a way to generalize all τ_n -atoms in the future.

8 Generalization to τ_n -atoms

To cut right to the chase, we were indeed able to generalize our findings in a satisfactory way:

Theorem 17. *Let n be an odd prime, and let y be an integer with a usual prime factorization of the form*

$$y = x_{01} * x_{02} * \dots * x_{0k} * x_{i1} * x_{i2} * \dots * x_{im} * x_j,$$

where $x_{ab} \in [x_a]$, $x_{ab} \neq n$ for all a, b , and $0 \neq i, j$.

Then y is a τ_n -atom if the following conditions hold:

1. if, for some integer z , $z \nmid m$, then $zi \not\equiv j \pmod{\frac{n-1}{2}}$,
2. if $mi + j \equiv 0 \pmod{\frac{n-1}{2}}$, then $k = 0$, and
3. $ci + j \not\equiv 0 \pmod{\frac{n-1}{2}}$ for any $c < m - 1$.

Proof. There are only finitely many ways in which y might fail to be an atom. If one were trying to properly τ_n -factor y , one might first try to create a proper τ_n -factorization involving terms already present in the usual prime factorization of y , namely x_0, x_i , or x_j factors.

Starting with the possibility of a proper τ_n -factorization involving all x_0 terms, clearly this would imply that $mi + j \equiv 0 \pmod{\frac{n-1}{2}}$, and so by condition 2 we see that $k = 0$; that is, there are no standard prime factors of y in the μ_n equivalence class $[x_0]$. Since $j \neq 0$, then, it must be necessary to take the product of x_j with some number of terms from $[x_i]$ in order to produce a single term in $[x_0]$, but condition 3 maintains that it will require at least $m - 1$ such terms. Thus, we are either left with a factorization of the form $x_0 * x_i$ or merely x_0 , and since $i \neq 0$ then both cases fail to produce a proper τ_n -factorization. Since a τ_n -factorization of terms in $[x_0]$ is impossible, then, we shall ignore all terms in $[x_0]$ henceforth, as they are absorbed into any arbitrary product.

Next, let us consider the possibility of a proper τ_n -factorization of terms all in $[x_i]$. Again, by condition 3, this case can quickly be discounted.

Finally, consider the possibility of a proper τ_n -factorization of terms all in x_j . Then clearly we must multiply sets of terms from $[x_i]$ in order to obtain at least one term from $[x_j]$. Suppose the minimum number of terms is d ; that is, if $ki \equiv j \pmod{\frac{n-1}{2}}$ then $k \geq d$. Due to condition 1, then, we know that $d \nmid m$; thus, there must be some leftover number of terms from $[x_i]$ which must produce an x_j term. Call this positive integer $c \neq d$. Suppose $c > d$. Then $c = d + r$ for some integer $r > 0$. Then we see that $j \equiv di \equiv ci \equiv (d+r)i \pmod{\frac{n-1}{2}}$, and so $di \equiv (d+r)i \pmod{\frac{n-1}{2}}$. Thus, $ri \equiv 0 \pmod{\frac{n-1}{2}}$. Suppose, then, that $r < d$. Then $(d-r)i \equiv di \equiv j \pmod{\frac{n-1}{2}}$; yet $d-r < d$, and d is the minimum number of x_i terms such that their product is equivalent to j , and so a contradiction arises. Thus, r must be greater than d . Let $r - d = s$. Then $ri \equiv (d+s)i \equiv 0 \pmod{\frac{n-1}{2}}$, and since $di \equiv j \pmod{\frac{n-1}{2}}$, then this means $si + j \equiv 0 \pmod{\frac{n-1}{2}}$. But certainly, since $s < d$ and $d < m$ then $s < m - 1$, and this is in violation of condition 3. Thus, we must conclude that $c < d$. Then, since d is the minimum number of x_i terms necessary to produce an x_j term, ci must be equivalent to $0 \pmod{\frac{n-1}{2}}$. Notice, though, that this implies that $di \equiv di - ci \equiv (d-c)i \equiv j$

$(\text{mod } \frac{n-1}{2})$, and clearly $d-c < d$; yet, again, d is the minimum number of x_i terms required to produce an x_j term. Again, a contradiction arises, and thus a τ_n -factorization of y involving terms only from $[x_j]$ must be impossible.

Therefore, if y is to have a proper τ_n -factorization, it must be composed of terms all from some μ_n equivalence class other than $[x_0]$, $[x_i]$, and $[x_j]$. Call this class $[x_g]$. Then suppose $y = x_{g1} * x_{g2} * \dots * x_{gz}$ for some positive integer z . Then $mi + j \equiv zg \pmod{\frac{n-1}{2}}$. Further, since condition 3 does not permit us to multiply x_j with some number of x_i terms to yield an x_0 term and still create such a factorization, we see that some product involving x_i terms and x_j terms must produce an x_g term; that is, $(m-a)i + j \equiv g \pmod{\frac{n-1}{2}}$ for some positive integer $a < m$. The remainder of the x_i terms must account for all the remaining x_g terms; that is, $ai \equiv (z-1)g \pmod{\frac{n-1}{2}}$. Now, as we are attempting to separate these a terms into $(z-1)$ groups, each of which are multiplied to yield a single x_g term, we have two options: either these $(z-1)$ groups all have the same number of x_i terms, or they do not. Suppose they do not; then select two groupings which have differing numbers of x_i terms, say $b > c$. Then $bi \equiv ci \equiv g \pmod{\frac{n-1}{2}}$. But then $(b-c)i \equiv 0 \pmod{\frac{n-1}{2}}$, and so we see that we can simply absorb $(b-c)$ terms from the larger grouping into a single x_0 term, and absorb that into our previous product of $(m-a)$ x_i terms and the single x_j term; in short, we will simply increase the value of a until we have $(z-1)$ groupings of an equal number x_i terms, each of which multiply to produce a single x_g term. Returning to our previous equivalence $ai \equiv (z-1)g \pmod{\frac{n-1}{2}}$, since the groupings of x_i factors have an equal number of terms, this implies that $(z-1)|a$. Thus $\frac{a}{z-1}$ must be an integer. Then $\frac{a}{z-1}i \equiv g \pmod{\frac{n-1}{2}}$; thus by transitivity $\frac{a}{z-1}i \equiv (m-a)i + j \pmod{\frac{n-1}{2}}$. Then $(m - \frac{az}{z-1})i + j \equiv 0 \pmod{\frac{n-1}{2}}$, so by condition 3 we see that $\frac{az}{z-1}$ must be either 1 or 0.

Case 1: $\frac{az}{z-1} = 0$. Then either $a = 0$ or $z = 0$, but both a and z are strictly positive; a contradiction arises.

Case 2: $\frac{az}{z-1} = 1$. Then $az = z - 1$, so $az < z$. This implies that $a < 1$. But a is a positive integer, so $a \geq 1$. Again, we see a contradiction.

Therefore, no τ_n -factorization of y can exist, and so y must be a τ_n -atom. \square

Looking back at our previous theorems, one can see that this generalization does indeed cover all τ_n -atoms mentioned in the paper other than the usual primes and integers of the form $np_1p_2\dots p_k$. We are pleased with this result, but more work is required before a complete generalization can be found. We stopped at $n = 11$ for a reason: $n = 13$ provides a truly difficult challenge, with atoms that can involve factors from at least 4 different μ_{13} equivalence classes. Moreover, our approach to τ_{11} involving the spreadsheet doesn't seem applicable to τ_{13} : while the τ_{11} sheet involved the more manageable number of 625 cases, τ_{13} would involve 7776 cases. We are very interested in the possibility of writing a program to generate and factor these cases far more efficiently, but until such a program is available to us we are left with the options of finding a new way to approach τ_{13} or simply enduring all those cases.

References

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